

CRITICAL POINTS OF RANDOM POLYNOMIALS WITH INDEPENDENT IDENTICALLY DISTRIBUTED ROOTS

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ABSTRACT. Let X_1, X_2, \dots be independent identically distributed random variables with values in \mathbb{C} . Denote by μ the probability distribution of X_1 . Consider a random polynomial $P_n(z) = (z - X_1) \dots (z - X_n)$. We prove a conjecture of Pemantle and Rivin [arXiv:1109.5975] that the empirical measure $\mu_n := \frac{1}{n-1} \sum_{P'_n(z)=0} \delta_z$ counting the complex zeros of the derivative P'_n converges in probability to μ , as $n \rightarrow \infty$.

1. STATEMENT OF THE RESULT

A critical point of a polynomial P is a root of its derivative P' . There are many results on the location of critical points of polynomials whose roots are known; see, e.g., [8]. One of the most famous examples is the Gauss–Lucas theorem stating that the complex critical points of any polynomial are located inside the convex hull of the complex zeros of this polynomial. Pemantle and Rivin [6] initiated the study of the probabilistic version of the problem. Let X_1, X_2, \dots be independent identically distributed (i.i.d.) random variables with values in \mathbb{C} . Denote by μ the probability distribution of X_1 . Consider a random polynomial

$$P_n(z) = (z - X_1) \dots (z - X_n).$$

Let μ_n be a probability measure which assigns to each critical point of P_n the same weight, that is

$$\mu_n = \frac{1}{n-1} \sum_{z \in \mathbb{C}: P'_n(z)=0} \delta_z.$$

We agree that the roots are always counted with multiplicities. Pemantle and Rivin [6] conjectured that the distribution of roots of P'_n should be stochastically close to the distribution of roots of P_n , for large n . In terms of logarithmic potentials, this means that the distribution of the equilibrium points of a two-dimensional electrostatic field generated by a large number of unit charges with i.i.d. locations should be close to the distribution of the charges themselves. More precisely, Pemantle and Rivin [6] conjectured that the random measure μ_n converges to μ in probability and gave a proof for all measures μ having a finite 1-energy. This restriction excludes some natural special cases, for example the uniform distribution on the circle. Our aim is to prove the conjecture in full generality.

Theorem 1.1. *Let μ be any probability measure on \mathbb{C} . Then, the sequence μ_n converges as $n \rightarrow \infty$ to μ in probability.*

2010 *Mathematics Subject Classification.* Primary, 30C15; secondary, 60G57, 60B10.

Key words and phrases. Random polynomials, empirical distribution, critical points, zeros of the derivative, logarithmic potential.

Few words about the mode of convergence. Let \mathbb{M} be the set of probability measures on \mathbb{C} . Endowed with the topology of weak convergence, \mathbb{M} becomes a Polish space. We view μ_n as a random element with values in \mathbb{M} and μ as a deterministic point in \mathbb{M} . With this convention, Theorem 1.1 states that for every open set $U \subset \mathbb{M}$ containing μ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\mu_n \notin U] = 0.$$

For example, Theorem 1.1 implies the following corollary: for every fixed $\varepsilon > 0$ the probability that at least $(1 - \varepsilon)n$ critical points of P_n are at distance at most ε from the support of the measure μ approaches 1, as $n \rightarrow \infty$. Since convergence in distribution and convergence in probability are equivalent if the limit is a.s. constant, see Lemma 3.7 in [4], we can state Theorem 1.1 as follows: the law of μ_n (viewed as a probability measure on \mathbb{M}) converges weakly to the unit point mass at μ .

Our proof is based on the connection with the logarithmic potential theory. The basic idea is to use the following formula (see, e.g., §2.4.1 in [1]): for every analytic function f (which does not vanish identically),

$$(1) \quad \frac{1}{2\pi} \Delta \log |f| = \sum_{z \in \mathbb{C}: f(z)=0} \delta_z.$$

Here, Δ is the Laplace operator which should be understood in the distributional sense. A similar method appeared in [3]; see also [2]. In [3] we studied polynomials whose coefficients (not roots) were independent random variables. Similar methods have been also used in the context of random matrix theory; see [9].

2. PROOF

2.1. Method of proof. Consider the logarithmic derivative of P_n :

$$(2) \quad L_n(z) := \frac{P'_n(z)}{P_n(z)} = \frac{1}{z - X_1} + \dots + \frac{1}{z - X_n}.$$

The main steps of the proof of Theorem 1.1 are collected in the following two lemmas.

Lemma 2.1. *There is a set $F \subset \mathbb{C}$ of Lebesgue measure 0 such that for every $z \in \mathbb{C} \setminus F$ we have*

$$(3) \quad \frac{1}{n} \log |L_n(z)| \xrightarrow[n \rightarrow \infty]{P} 0.$$

Lemma 2.2. *Let λ be the Lebesgue measure on \mathbb{C} and $\psi : \mathbb{C} \rightarrow \mathbb{R}$ any compactly supported continuous function. Then,*

$$(4) \quad \frac{1}{n} \int_{\mathbb{C}} (\log |L_n(z)|) \psi(z) d\lambda(z) \xrightarrow[n \rightarrow \infty]{P} 0.$$

After the lemmas have been established, the proof of Theorem 1.1 can be completed as follows. It suffices to show that for every infinitely differentiable, compactly supported function $\varphi : \mathbb{C} \rightarrow \mathbb{R}$,

$$(5) \quad \frac{1}{n} \sum_{z \in \mathbb{C}: P'_n(z)=0} \varphi(z) \xrightarrow[n \rightarrow \infty]{P} \int_{\mathbb{C}} \varphi d\mu.$$

Indeed, (5) implies that the law of μ_n converges weakly (as a probability measure on \mathbb{M}) to the unit point mass at μ ; see Theorem 14.16 in [4]. This implies convergence in probability since the limit is constant a.s. To prove (5) we use the formula

$$(6) \quad \frac{1}{2\pi n} \int_{\mathbb{C}} (\log |L_n(z)|) \Delta \varphi(z) d\lambda(z) = \frac{1}{n} \sum_{z \in \mathbb{C}: P'_n(z)=0} \varphi(z) - \frac{1}{n} \sum_{z \in \mathbb{C}: P_n(z)=0} \varphi(z).$$

It follows from (1) with $f = P'_n$ and $f = P_n$ after subtraction and division by n . As $n \rightarrow \infty$, the left-hand side of (6) tends to 0 in probability by Lemma 2.2. Since the zeros of P_n are i.i.d. random variables, the second term in the right-hand side of (6) tends to $\int \varphi d\mu$ in probability (and even a.s.) by the law of large numbers. This proves (5). In the rest of the paper we are occupied with the proofs of Lemmas 2.1 and 2.2. Let $\mathbb{D}_r(z) = \{x \in \mathbb{C} : |x - z| < r\}$ be the disk of radius $r > 0$ centered at $z \in \mathbb{C}$ and $\bar{\mathbb{D}}_r(z)$ its closure. We also write $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\bar{\mathbb{D}}_r = \bar{\mathbb{D}}_r(0)$.

2.2. Proof of Lemma 2.1. First of all, let us stress that in general, (3) does not hold for every $z \in \mathbb{C}$ since it evidently fails if z is an atom of μ . We need to introduce an exceptional set F . It consists of points at which μ has bad regularity properties. Let

$$\log_- z = \begin{cases} |\log z|, & 0 \leq z \leq 1, \\ 0, & z \geq 1, \end{cases} \quad \log_+ z = \begin{cases} 0, & 0 \leq z \leq 1, \\ \log z, & z \geq 1. \end{cases}$$

Note that $\log_- 0 = +\infty$.

Lemma 2.3. *Let $F = \{z \in \mathbb{C} : \int_{\mathbb{C}} \log_- |y - z| d\mu(y) = \infty\}$. Then, the Lebesgue measure of F is zero.*

Proof. Recall that λ is the Lebesgue measure on \mathbb{C} . We have, by Fubini's theorem,

$$\int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log_- |y - z| d\mu(y) \right) d\lambda(z) = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log_- |z - y| d\lambda(z) \right) d\mu(y) = \frac{\pi}{2},$$

where the second equality holds since the integral in the brackets is $\pi/2$ for every $y \in \mathbb{C}$ and μ is a probability measure. It follows that $\lambda(F) = 0$. \square

Lemma 2.4. *For every $z \in \mathbb{C} \setminus F$ we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |L_n(z)| \leq 0$ a.s.*

Corollary 2.5. *For every $z \in \mathbb{C} \setminus F$ and every $\varepsilon > 0$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \log |L_n(z)| \geq \varepsilon \right] = 0.$$

Proof of Lemma 2.4. The idea is to show that the poles of L_n do not approach z too fast. Fix $\varepsilon > 0$. We have

$$\int_{\mathbb{C}} \log_- |y - z| d\mu(y) \geq \varepsilon \sum_{n=1}^{\infty} \mu(\mathbb{D}_{e^{-\varepsilon n}}(z)).$$

Since the left-hand side is finite by the assumption $z \notin F$, the right-hand side must be finite, too. It follows that

$$\sum_{n=1}^{\infty} \mathbb{P} \left[\frac{1}{|z - X_n|} > e^{\varepsilon n} \right] = \sum_{n=1}^{\infty} \mu(\mathbb{D}_{e^{-\varepsilon n}}(z)) < \infty.$$

By the Borel–Cantelli lemma, we have $\frac{1}{|z-X_n|} \leq e^{\varepsilon n}$ for all but finitely many n . Also, z is not an atom of μ (since $z \notin F$) and hence, $X_n \neq z$ for all $n \in \mathbb{N}$ a.s. It follows that there is an a.s. finite random variable M such that

$$|L_n(z)| \leq M + ne^{\varepsilon n} \leq e^{2\varepsilon n},$$

where the second inequality holds for large n . Thus, $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |L_n(z)| \leq 2\varepsilon$. Since this holds for every $\varepsilon > 0$, the proof is completed. \square

Lemma 2.6. *For every $z \in \mathbb{C}$ and every $\varepsilon > 0$, we have*

$$(7) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \log |L_n(z)| \leq -\varepsilon \right] = 0.$$

Proof. If $X_i = c$ a.s., then $L_n(z) = n/(z-c)$ and the lemma holds trivially. Assume therefore that the X_i 's are non-degenerate. Given a real-valued random variable ξ we denote by

$$Q(\xi; \delta) = \sup_{t \in \mathbb{R}} \mathbb{P}[t \leq \xi \leq t + \delta], \quad \delta > 0,$$

the concentration function of ξ . We will use the fact that the concentration function of the sum of n i.i.d. random variables decays like $O(1/\sqrt{n})$. More precisely, by Theorem 2.22 on p. 76 in [7] there is an explicit absolute constant C such that for every sequence of non-degenerate i.i.d. real-valued random variables ξ_1, ξ_2, \dots and for all $n \in \mathbb{N}$, $\delta > 0$, we have

$$(8) \quad Q(\xi_1 + \dots + \xi_n; \delta) \leq C \frac{1 + \delta}{\sqrt{n}}.$$

Note that no moment requirements on the ξ_i 's are imposed. If $z \in \mathbb{C}$ is an atom of μ , then (7) holds trivially since $|L_n(z)| = \infty$ with probability approaching 1 as $n \rightarrow \infty$. Fix $z \in \mathbb{C}$ which is not an atom of μ . Consider the complex-valued random variables $Y_i = \frac{1}{z-X_i}$, $i \in \mathbb{N}$. They are non-degenerate since we assume that the X_i 's are non-degenerate. It follows that at least one of the random variables $\operatorname{Re} Y_1$ or $\operatorname{Im} Y_1$ is non-degenerate. Suppose for concreteness that $\operatorname{Re} Y_1$ is non-degenerate. Then,

$$\mathbb{P}[|L_n(z)| \leq e^{-\varepsilon n}] \leq \mathbb{P} \left[\left| \sum_{k=1}^n \operatorname{Re} Y_k \right| \leq e^{-\varepsilon n} \right] \leq Q \left(\sum_{k=1}^n \operatorname{Re} Y_k, 2e^{-\varepsilon n} \right) \leq \frac{2C}{\sqrt{n}}.$$

The last inequality follows from (8) for n large. This completes the proof. \square

Combining Corollary 2.5 and Lemma 2.6 we obtain Lemma 2.1.

2.3. Proof of Lemma 2.2. We already know from Lemma 2.1 that $\frac{1}{n} \log |L_n(z)|$ converges to 0 in probability for Lebesgue almost all $z \in \mathbb{C}$. To prove Lemma 2.2 we need to interchange the limit and the integral in (4). This is done by means of the following lemma whose proof can be found in [9].

Lemma 2.7 (Lemma 3.1 in [9]). *Let (X, \mathcal{A}, ν) be a finite measure space and $f_1, f_2, \dots : X \rightarrow \mathbb{R}$ random functions which are defined over a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and are jointly measurable with respect to $\mathcal{A} \otimes \mathcal{B}$. Assume that:*

- (1) *For ν -a.e. $x \in X$ we have $f_n(x) \rightarrow 0$ in probability, as $n \rightarrow \infty$.*
- (2) *For some $\delta > 0$, the sequence $\int_X |f_n(x)|^{1+\delta} d\nu(x)$ is tight.*

Then, $\int_X f_n(x) d\nu(x)$ converges in probability to 0.

Recall that ψ is a continuous function with compact support. Let r be such that the support of ψ is contained in the disk \mathbb{D}_r . The first condition of Lemma 2.7, with $f_n(z) = \frac{1}{n}(\log |L_n(z)|)\psi(z)$, $X = \mathbb{D}_r$, and $\nu = \lambda$ has been already verified in Lemma 2.1. The second condition with $\delta = 1$ follows from the next lemma.

Lemma 2.8. *The sequence $\frac{1}{n^2} \int_{\mathbb{D}_r} \log^2 |L_n(z)| d\lambda(z)$ is tight.*

The rest of the paper is devoted to the proof of Lemma 2.8. First we need to prove a statement which is a uniform version of Lemma 2.4. This statement implies Lemma 2.4, but for clarity, we stated Lemma 2.4 separately. For $R > 0$ define

$$(9) \quad M_n(R) = \sup_{|z|=R} |L_n(z)|.$$

Lemma 2.9. *There is a set $E \subset (0, \infty)$ of Lebesgue measure 0 such that for every $R \in (0, \infty) \setminus E$ we have*

$$(10) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log M_n(R) \leq 0 \quad a.s.$$

Proof. The proof is similar to that of Lemmas 2.3 and 2.4. Let $\bar{\mu}$ be the radial part of μ . This means that $\bar{\mu}$ is a measure on $[0, \infty)$ defined by $\bar{\mu}([0, s)) = \mu(|z| < s)$ for all $s > 0$. Define a set $E = \{R > 0 : \int_{\mathbb{R}} \log_- |x - R| d\bar{\mu}(x) = \infty\}$. By Fubini's theorem,

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \log_- |x - R| d\bar{\mu}(x) \right) dR = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \log_- |R - x| dR \right) d\bar{\mu}(x) = 2$$

since $\int_{\mathbb{R}} \log_- |R - x| dR = 2$ for every $x \in \mathbb{R}$ and $\bar{\mu}$ is a probability measure. Hence, $\lambda(E) = 0$. We now take $R \in (0, \infty) \setminus E$ and prove (10). Fix $\varepsilon > 0$. We have

$$\int_{\mathbb{R}} \log_- |x - R| d\bar{\mu}(x) \geq \varepsilon \sum_{n=1}^{\infty} \bar{\mu}((R - e^{-\varepsilon n}, R + e^{-\varepsilon n})) = \varepsilon \sum_{n=1}^{\infty} \mu(\mathbb{D}_{R+e^{-\varepsilon n}} \setminus \bar{\mathbb{D}}_{R-e^{-\varepsilon n}}).$$

The left-hand side is finite by the assumption $R \notin E$, hence the right-hand side is finite, too. It follows that

$$\sum_{n=1}^{\infty} \mathbb{P} \left[\sup_{|z|=R} \frac{1}{|z - X_n|} > e^{\varepsilon n} \right] = \sum_{n=1}^{\infty} \mu(\mathbb{D}_{R+e^{-\varepsilon n}} \setminus \bar{\mathbb{D}}_{R-e^{-\varepsilon n}}) < \infty.$$

By the Borel–Cantelli lemma, we have $\sup_{|z|=R} \frac{1}{|z - X_n|} \leq e^{\varepsilon n}$ for all but finitely many n . Note that R is not an atom $\bar{\mu}$ (since $R \notin E$) and therefore, $|X_n| \neq R$ for all $n \in \mathbb{N}$ a.s. Hence, there is an a.s. finite random variable M such that

$$M_n(R) \leq \sum_{k=1}^n \sup_{|z|=R} \frac{1}{|z - X_k|} \leq M + ne^{\varepsilon n} \leq e^{2\varepsilon n},$$

for large n . In the first inequality we used (2) and (9). Since $\varepsilon > 0$ is arbitrary, the proof is completed. \square

To prove Lemma 2.8 we need to control the zeros and the poles of L_n since at these points $\log |L_n(z)|$ becomes infinite. We will use the Poisson–Jensen formula. Take some $R > r$. Denote by $x_{1,n}, \dots, x_{k_n,n}$ those zeros of P_n which are located in the disk \mathbb{D}_R . They form a subset of X_1, \dots, X_n . Let also $y_{1,n}, \dots, y_{l_n,n}$ be the zeros of P'_n located in the disk \mathbb{D}_R . Note that $k_n \leq n$ and $l_n < n$. By the Poisson–Jensen

formula, see [5, Chapter 8], we have for any $z \in \mathbb{D}_R$ which is not a zero or pole of L_n ,

$$(11) \quad \log |L_n(z)| = I_n(z; R) + \sum_{l=1}^{l_n} \log \left| \frac{R(z - y_{l,n})}{R^2 - \bar{y}_{l,n}z} \right| - \sum_{k=1}^{k_n} \log \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n}z} \right|,$$

where we use the notation

$$(12) \quad I_n(z; R) = \frac{1}{2\pi} \int_0^{2\pi} \log |L_n(Re^{i\theta})| P_R(|z|, \theta - \arg z) d\theta$$

and P_R is the Poisson kernel

$$(13) \quad P_R(\rho, \varphi) = \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \varphi}, \quad \rho \in [0, R], \varphi \in [0, 2\pi].$$

Lemma 2.10. *There is $R > 2r$ such that we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{z \in \mathbb{D}_r} I_n(z; R) \leq 0 \quad a.s.$$

Corollary 2.11. *There is $R > 2r$ such that for every $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \sup_{z \in \mathbb{D}_r} I_n(z; R) \geq \varepsilon \right] = 0.$$

Proof of Lemma 2.10. Choose any $R > 2r$ not contained in the exceptional set E of Lemma 2.9. It follows from (13) that there is $C = C(r, R)$ such that $0 < P_R(|z|, \theta) < C$ for all $z \in \mathbb{D}_r$ and $\theta \in [0, 2\pi]$. It follows from (9) and (12) that $I_n(z; R) \leq C \log M_n(R)$ for all $z \in \mathbb{D}_r$. The proof is completed by using Lemma 2.9. \square

In the sequel we choose $R \in (2r, \infty) \setminus E$ as in the above proof. In the next two lemmas we establish a lower bound for $I_n(z; R)$ which is uniform in $z \in \mathbb{D}_r$. First we consider the case $z = 0$. Recall that F is an exceptional set defined in Lemma 2.3.

Lemma 2.12. *Assume that $0 \notin F$. There is a constant $A = A(R)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} I_n(0; R) \leq -A \right] = 0.$$

Proof. In the special case $z = 0$ the Poisson–Jensen formula (11) takes the form

$$(14) \quad \frac{1}{n} I_n(0; R) = \frac{1}{n} \log |L_n(0)| - \frac{1}{n} \sum_{l=1}^{l_n} \log \left| \frac{y_{l,n}}{R} \right| + \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right|.$$

Recall that $x_{1,n}, \dots, x_{k_n,n}$ are those of the points X_1, \dots, X_n which belong to the disk \mathbb{D}_R . By the law of large numbers,

$$(15) \quad \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right| \xrightarrow[n \rightarrow \infty]{a.s.} -\mathbb{E} \left[\log_- \left| \frac{X_1}{R} \right| \right].$$

The expectation on the right-hand side is finite. To see this note that $z \mapsto \log_- |z/R| - \log_- |z|$ is a bounded function with compact support and recall that $\mathbb{E} \log_- |X_1| < \infty$ by the assumption $0 \notin F$. It follows from Lemma 2.6 and (15) that there is $A_1 = A_1(R)$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \log |L_n(0)| \leq -1 \text{ or } \frac{1}{n} \sum_{k=1}^{k_n} \log \left| \frac{x_{k,n}}{R} \right| \leq -A_1 \right] = 0.$$

For the second term on the right-hand side of (14) we have trivially

$$\frac{1}{n} \sum_{l=1}^{l_n} \log \left| \frac{y_{l,n}}{R} \right| \leq 0.$$

The statement of the lemma follows with $A = A_1 + 1$. \square

In the sequel we assume that $0 \notin F$. This is not a restriction of generality since in the case $0 \in F$ we can choose any $a \notin F$ (which exists by Lemma 2.3) and prove Theorem 1.1 for the random variables $Y_i = X_i - a$ instead of X_i .

Lemma 2.13. *There is a constant $B = B(r, R)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \inf_{z \in \mathbb{D}_r} I_n(z; R) \leq -B \right] = 0.$$

Proof. Write $q_n^+(\theta) = \frac{1}{n} \log_+ |L_n(Re^{i\theta})|$ and $q_n^-(\theta) = \frac{1}{n} \log_- |L_n(Re^{i\theta})|$, where $\theta \in [0, 2\pi]$. Then, $q_n(\theta) := \frac{1}{n} \log |L_n(Re^{i\theta})| = q_n^+(\theta) - q_n^-(\theta)$. Note that $q_n^+(\theta) \geq 0$ and $q_n^-(\theta) \geq 0$. By (13) and the assumption $R > 2r$ there is a constant $C = C(r, R) > 1$ such that $1/C < P_R(|z|, \theta) < C$ for all $z \in \mathbb{D}_r$, $\theta \in [0, 2\pi]$. It follows that for $z \in \mathbb{D}_r$,

$$\begin{aligned} \frac{2\pi}{n} I_n(z; R) &= \int_0^{2\pi} q_n^+(\theta) P_R(|z|, \theta - \arg z) d\theta - \int_0^{2\pi} q_n^-(\theta) P_R(|z|, \theta - \arg z) d\theta \\ &\geq \frac{1}{C} \int_0^{2\pi} q_n^+(\theta) d\theta - C \int_0^{2\pi} q_n^-(\theta) d\theta \\ &= \frac{2\pi C}{n} I_n(0; R) - \left(C - \frac{1}{C} \right) \int_0^{2\pi} q_n^+(\theta) d\theta \\ &\geq \frac{2\pi C}{n} I_n(0; R) - 2\pi \left(C - \frac{1}{C} \right) \frac{1}{n} \log M_n(R). \end{aligned}$$

We have used that $I_n(0; R) = \frac{1}{2\pi} \int_0^{2\pi} \log |L_n(Re^{i\theta})| d\theta$. By Lemma 2.12 and Lemma 2.9 (recall that $R \notin E$) we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} I_n(0; R) \leq -A \text{ or } \frac{1}{n} \log M_n(R) > 1 \right] = 0.$$

The statement of the lemma follows. \square

We are in position to complete the proof of Lemma 2.8. Applying the inequality between the arithmetic and quadratic means several times to the Poisson–Jensen formula (11) and dividing by n^2 we obtain

$$\begin{aligned} (16) \quad & \frac{1}{n^2} \log^2 |L_n(z)| \\ & \leq \frac{3}{n^2} I_n^2(z; R) + \frac{3l_n}{n^2} \sum_{l=1}^{l_n} \log^2 \left| \frac{R(z - y_{l,n})}{R^2 - \bar{y}_{l,n}z} \right| + \frac{3k_n}{n^2} \sum_{k=1}^{k_n} \log^2 \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n}z} \right|. \end{aligned}$$

It follows from Corollary 2.11 and Lemma 2.13 that the sequence $\frac{3}{n^2} \int_{\mathbb{D}_r} I_n^2(z; R) d\lambda(z)$ is tight. We estimate the remaining two terms in the right-hand side of (16). We have, for some finite $C = C(r, R)$,

$$\sup_{y \in \mathbb{D}_R} \int_{\mathbb{D}_r} \log^2 \left| \frac{R(z - y)}{R^2 - \bar{y}z} \right| d\lambda(z) < C.$$

To see this, note that $|R^2 - \bar{y}z|$ remains bounded below as long as $z \in \mathbb{D}_r$, $y \in \mathbb{D}_R$. and use the integrability of the squared logarithm. Recall also that k_n (resp., l_n) is the number of roots of P_n (resp., P'_n) in the disk \mathbb{D}_R . Hence, both numbers do not exceed n . It follows that there is a deterministic constant $C_1 = C_1(r, R)$ such that for every $n \in \mathbb{N}$,

$$\frac{3l_n}{n^2} \sum_{l=1}^{l_n} \int_{\mathbb{D}_r} \log^2 \left| \frac{R(z - y_{l,n})}{R^2 - \bar{y}_{l,n}z} \right| d\lambda(z) + \frac{3k_n}{n^2} \sum_{k=1}^{k_n} \int_{\mathbb{D}_r} \log^2 \left| \frac{R(z - x_{k,n})}{R^2 - \bar{x}_{k,n}z} \right| d\lambda(z) \leq C.$$

The sum of a tight sequence and an a.s. bounded sequence is tight. Hence, the sequence $\frac{1}{n^2} \int_{\mathbb{D}_r} \log^2 |L_n(z)| d\lambda(z)$ is tight. The proof of Lemma 2.8 is complete.

Remark 2.14. We conjecture that Theorem 1.1 is valid in the sense of a.s. convergence. The whole proof can be easily adapted to yield the a.s. convergence, except for Lemma 2.6. To see the difficulty, consider the Bernoulli case in which X_i takes the values ± 1 with probability $1/2$. Then, there are a.s. infinitely many times n with $L_n(0) = 0$ and hence, $\frac{1}{n} \log |L_n(0)|$ does not converge to 0 a.s., even though it converges to 0 in probability. The difficulty in proving the a.s. convergence is thus to show that there are not too many points $z \in \mathbb{C}$ for which strong cancellation among the terms $\frac{1}{z - X_i}$ is possible. It is possible to prove the a.s. analogue of Lemma 2.6 under additional regularity assumptions on the distribution of X_i (for example, existence of bounded density).

Acknowledgment. The author is grateful to D. Zaporozhets for numerous discussions on the topic of the paper.

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